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EXPANSION INTO VACUUM OF A GASEOUS ELLIPSOID WITH REPULSION BETWEEN THE PARTICLES*

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Methods of the analytic theory of differential equations as well as the numerical methods are used to study the problem of adiabatic expansion of a homogeneous cloud of perfect gas into vacuum. The Cartesian and Lagrangian coordinates of the gas particles are connected by a linear relation, and the particles have a charge which generates a macroscopic field within the cloud, without however affecting the properties of the perfect gas.

The problem of expansion of an inviscid gas cloud with the ellipsoidal level surfaces is well known and has a number of solutions such as that due to L.I. Sedov /1/, solution of the problem of gravitating gas /2/ and solution of a particular problem of adiabatic expansion of a cloud in the presence of a constraint of the type

$$x_i = \sum_k F_{ik} a_k$$

connecting the Lagrangian a_i and Cartesian x_i coordinates of the particles /3/. The method of /3/ leads to a simple system of ordinary differential equations for the quantities F_{ik}, and the system was solved analytically for a number of nontrivial cases /4/. Using the formulation of /3/, the problem can be generalized to the case when a gravitational attraction exists between the particles and the gas density is constant over the volume of the cloud /5/. Here the author uses the methods of analytic theory of differential equations to carry out a detailed investigation of the possible transitions of the ellipsoid between the states determined by the set of singular points of the corresponding system of equations. Thus the solution /6/ of the problem of motion of a gravitating dust cloud and solutions on the submanifolds with fixed values of certain variables, were used to study the motions corresponding to oscillatory disintegration of an ellipsoid and being of interest in astrophysics. The case in which the particles experience repulsion instead of attraction corresponds to the motion of a gas cloud with particles carrying the same charge. Such a problem arises in plasma physics and in the corresponding branches of astrophysics, in particular in the study of the processes taking place near the centers of active cosmic objects such as Seifert galaxies and quasars.

The following initial equations are used in our study of the motion of gas. The equations of continuity and impulse

$$\frac{d\rho}{dt} + \rho \sum \frac{\partial v_i}{\partial x_i} = 0, \quad \rho \frac{dv_i}{dt} = -\frac{\partial P}{\partial x_i} - \rho \frac{q}{m} \frac{\partial \Phi}{\partial x_i} \tag{1}$$

and the expression for the potential generated by the ellipsoid charged uniformly over the volume, and the internal points of the ellipsoid

$$\Phi = \frac{\alpha_2 m}{2q} \int_0^\infty \left(1 - \frac{x_1'^2}{d_1^2+s} - \frac{x_2'^2}{d_2^2+s} - \frac{x_3'^2}{d_3^2+s}\right) \times \frac{ds}{[(d_1^2+s)(d_2^2+s)(d_3^2+s)]^{1/2}} \tag{2}$$

Here d_i denote the semi-axes of the gaseous ellipsoid, α_2 is a coefficient depending on the cloud charge density, q and m are the charge and mass of the gas particles respectively, and x_i' are the coordinates of a point in the coordinate system whose axes coincide with the directions of the semi-axes.

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We assume that the gas density is constant over the cloud volume up to its boundary, while the temperature and pressure and quadratic functions of the coordinates with a maximum at the cloud center. The gas is assumed perfect and the process of expansion (compression) adiabatic, with the adiabatic index γ, i.e. $d\,(P/\rho^{\gamma})/dt = 0$. The gas particles possess a charge of equal sign and strength, creating a macroscopis field within the cloud. We shall assume that this does not imply the necessity of allowing, in the local thermodynamic formulation, for the deviations of the gas from perfect, i.e. the condition $n \ll (kT/q^2)^3$ where n is the particle number density, k is the Boltzmann constant and T is the gas temperature, holds. We assume that the following relation holds for the gas motion:

$$x_i = \sum_k F_{ik} a_k \tag{3}$$

where x_i are the Cartesian coordinates of the gas sections, a_k are the Lagrangian coordinates and F_{ik} are functions of time. From (1)—(3) we obtain the following system of equations:

$$\ddot{F}_{ik} = \alpha_1 (\gamma - 1) F_{ki}^{-1} + \alpha_2 \int_0^\infty \sum_n \left(\frac{L_{1i}L_{1n}}{d_1^2 + s} + \frac{L_{2i}L_{2n}}{d_2^2 + s} + \frac{L_{3i}L_{3n}}{d_3^2 + s} \right) \times \frac{F_{nk}\, ds}{[(d_1^2+s)(d_2^2+s)(d_3^2+s)]^{1/2}} \tag{4}$$

$$P = \frac{3M(\gamma-1)(1 - a^2/a_0^2)\, W \varphi_0^{\gamma-1}}{4\pi a_0^3 \varphi^{\gamma}}, \quad \varphi = \det \| F_{ik} \|$$

$$\varphi_0 = \varphi(t = 0), \quad \rho = \frac{3M}{4\pi a_0^3 \varphi}, \quad \alpha_1 = \frac{2W\varphi_0^{\gamma-1}}{a_0^2}, \quad \alpha_2 = 2\pi\rho_0\varphi_0 a_0^3 \left(\frac{q}{m}\right)^2$$

Here L_{ik} is the rotation matrix which superimposes the coordinate axes on the directions of the ellipsoid semi-axes, M is the mass of the gas cloud and W is the specific energy of the gas at the initial instant at the cloud center. In the course of deriving (4) we have utilized the first law of thermodynamics, and the above equations can be written in the form of the Hamilton—Jacobi equations where the Hamiltonian is given by the expression

$$H = \frac{1}{2} \sum_i p_i^2 + \alpha_1 V(q)^{1-\gamma} + \alpha_2 U(q)$$

$$\{p_i\} = \{\dot{F}_{ik}\}, \quad \{q_i\} = \{F_{ik}\}, \quad V(q) = \varphi(F_{ik})$$

$$U(q) = \int_0^\infty \frac{ds}{[(d_1^2+s)(d_2^2+s)(d_3^2+s)]^{1/2}}$$

We analyze the solutions of the system of equations (4) using the methods of the analytic theory of differential equations. The Hamiltonian of the system in question is always positive and there are no solutions of the problem with negative energy. In the coordinates W_1:

$$p_i^* = \frac{p_i}{(\alpha_1 V(q)^{1-\gamma} + \alpha_2 U(q))^{1/2}}, \quad u = \frac{U(q)}{\alpha_1\alpha_2^{-1} V(q)^{1-\gamma} + U(q)}$$

$$y_i = \frac{q_i}{q_0}, \quad \frac{d\tau_1}{dt} = \frac{(\alpha_1 V(q)^{1-\gamma} + \alpha_2 U(q))^{1/2}}{q_0 V(q)}, \quad q_0 = \Big(\sum_k q_k^2\Big)^{1/2}$$

the system (4) assumes the form

$$\begin{aligned} \frac{dy_i}{d\tau_1} &= V\Big(p_i^* - y_i \sum y_k p_k^*\Big) \\ \frac{du}{d\tau_1} &= u(1-u)\sum_i p_i^* \left[\frac{V}{U}\frac{\partial U}{\partial y_i} + (\gamma-1)\frac{\partial V}{\partial y_i}\right] \\ \frac{dp_i^*}{d\tau_1} &= \sum_k \left(\delta_{ik} + \frac{p_i^* p_k^*}{2}\right)\left[(\gamma-1)(1-u)\frac{\partial V}{\partial y_k} - u\frac{V}{U}\frac{\partial U}{\partial y_k}\right] \end{aligned} \tag{5}$$

and in coordinates W_2:

$$P_i = \frac{p_i}{p_0}, \quad u = \frac{U(q)}{\alpha_1\alpha_2^{-1} V(q)^{1-\gamma} + U(q)}, \quad y_i = \frac{q_i}{q_0}$$

$$w = \frac{\alpha_1 V(q)^{1-\gamma} + \alpha_2 U(q)}{p_0^2}, \quad \frac{d\tau_2}{dt} = \frac{p_0}{q_0 V(y)}, \quad p_0 = \Big(\sum_k p_k^2\Big)^{1/2}$$

the system (4) becomes

$$\frac{dy_i}{d\tau_2} = V\left(P_i - y_i \sum_k y_k P_k\right) \tag{6}$$
$$\frac{du}{d\tau_2} = u(1-u)\sum_k P_k\left[\frac{V}{U}\frac{\partial U}{\partial y_k} + (\gamma-1)\frac{\partial V}{\partial y_k}\right]$$
$$\frac{dw}{d\tau_2} = w(1+2w)\sum_k P_k\left[(1-\gamma)(1-u)\frac{\partial V}{\partial y_k} + u\frac{V}{U}\frac{\partial U}{\partial y_k}\right]$$
$$\frac{dP_i}{d\tau_2} = w\sum_k(\delta_{ik} - P_i P_k)\left[(\gamma-1)(1-u)\frac{\partial V}{\partial y_k} - u\frac{V}{U}\frac{\partial U}{\partial y_k}\right]$$

Let us introduce the notation

$$V_p{}^* = \sum_i p_i{}^* \frac{\partial V}{\partial y_i}, \quad \varepsilon = \mathrm{sgn}(V_p{}^*)$$

and analyze the singularities of the systems (5) and (6), retaining the notation(*). The condition $V = 0$, $u = 1$ determines the singular points K_ε. When $V = 0$, the difference between the ellipsoid semi-axes is considerable. The condition $u = 1$ corresponds to large volume of the cloud, or to the potential energy in the electric field exceeding considerably the energy of thermal motion of the particles. The case of positive ε corresponds to an increase in the cloud volume, and negative ε to a decrease in the volume. The points K_{-1} have an incoming separatrix for which $u = 1$, and the outgoing separatrix for which $V = 0$. The change in the sign of ε converts the incoming separatrix into the outgoing one, and vice versa. From the physical point of view it is clear that the system in question with repulsion cannot change from expansion to compression, therefore only one branch of transitions $K_{-1} \to K_1$ is possible which is not repeated after its first appearance. For the points K_ε we have two eigenvalues of the form $V_p{}^*$, $(1-\gamma)V_p{}^*$, and the separatrices mentioned above correspond to these eigenvalues. Let us introduce the notation

$$V_p = \sum_i P_i \frac{\partial V}{\partial y_i}, \quad \beta = \mathrm{sgn}(V_p)$$

We have physically interesting singularities $M_{0,\beta}$, $M_{1,\beta}$, $N_{0,\beta}$, $N_{1,\beta}$ in the coordinates W_2. The singularities $N_{1,\beta}$ ($V = 0$, $u = 1$, $w = 0$) have two nonzero eigenvalues: $(1-\gamma)V_p$, V_p. The case $w = 0$ corresponds to the kinetic energy of the motion of gas exceeding considerably the potential energy in the electric field and the energy of thermal motion of the particles. This is possible in the case of unbounded inertial expansion, or when the initial motion of gas with nonzero velocity takes place from infinity towards the center. The above eigenvalues have different signs. For $\beta = 1$ we have the incoming separatrix with λ_1, and the outgoing separatrix with λ_2. When $\beta = -1$, the separatrices interchange their positions, and for λ_1 we have $V = 0$, $w = 0$, and $\lambda_2 - u = 1$, $w = 0$.

The singularties $N_{0,\beta}$ ($w = 0$, $u = 0$, $V = 0$) have eigenvalues $(\gamma-1)V_p$, $(1-\gamma)V_p$, V_p. When $\beta > 0$, this yields one incoming and two outgoing separatrices, and for $\beta < 0$ it is the other way round. The condition $u = 0$ corresponds either to the energy of thermal particle motion exceeding their energy in the electric field, or to a very small volume of the cloud. For the separatrix with the first eigenvalue we have $w = 0, V = 0$, for the second $u = 0$, $V = 0$ and for the third $w = 0$, $u = 0$. The singularities $M_{0,\beta}$ ($P_i = \beta y_i$, $w = 0$, $u = 0$) have the eigenvalues $-3(\gamma-1)\beta V$, $(3\gamma-4)\beta V$, $-\beta V$, $-2\beta V$. When $P_i = \beta y_i$, the macroscopic rotational motion of the gas is absent. The set of eigenvalues is such, that both incoming and outgoing separatrices are present. When $\gamma > 4/3$ $M_{0,1}$, has one incoming and $M_{0,-1}$ one outgoing separatrix. When $\gamma < 4/3$ $M_{0,1}$ has not outgoing separatrices and is attractive, while $M_{0,-1}$ has no incoming separatrices and is repulsive. The incoming separatrix of the point $M_{0,1}$ occurs, at $\gamma > 4/3$, when $P_i = \beta y_i$, $w = 0$. The singularities $M_{1,\beta}$ ($P_i = \beta y_i$, $w = 0$, $u = 1$) have the eigenvalues $\beta V(4-3\gamma)$, $-\beta V$, $-2\beta V$. When $\gamma > 4/3$, point $M_{1,1}$ has not outgoing separatrices and is attractive, while $M_{1,-1}$ has no incoming separatrices and is repulsive. When $\gamma < 4/3$, the point $M_{1,1}$

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*)Bogoiavlenskii O.I. Oscillatory mode of expansion of a gaseous cloud into vacuum. Preprint In-ta teoreticheskoi fiziki Akad. Nauk SSSR, Chernogolovka, 1975.

has one outgoing and $M_{1,-1}$ one incoming separatrix, and we have for these separatrices $w = 0$, $P_i = \beta y_i$. Thus on passing through the value $\gamma = {}^4/_3$ the parts played by the points $M_{1,\beta}$ and $M_{0,\beta}$ are interchanged. The points $M_{1,1}$ correspond to a more or less uniform expansion along all three axes, and the points $M_{0,1}$ correspond to an expansion with pronounced ellipticity.

Solution of the systems of equations on the submanifolds $V = 0$, $w = 0$ $(u = 0,\ V = 0)$ can be carried out in the same manner (*). The solution on the submanifold $u = 1$ should correspond to the expansion of a charged dust cloud. For such a system the impossiblity of a changeover from expansion to contraction is valid, since there are no forces which could force the gas to move back towards the center.

Let us give the possible transitions between the singularities of the system along the separatrices. The scheme of transitions for the case $\gamma > {}^4/_3$ is given in Fig.1.

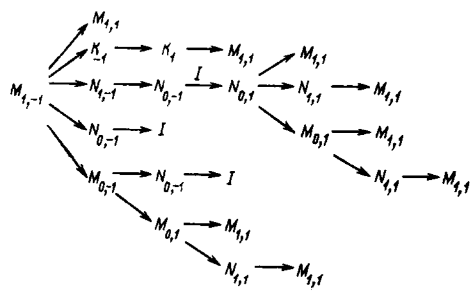

Fig.1

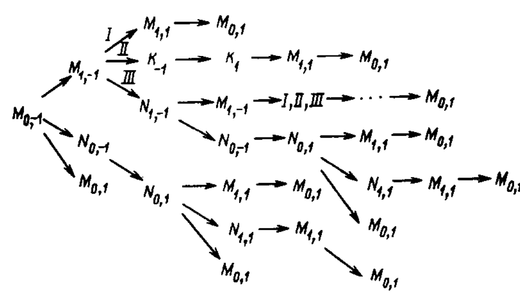

Fig.2

The characteristic feature of this system is, that it does not contain repeated links. The motion can originate at any position of the scheme to end always at the point $M_{1,1}$. Fig.2 depicts the scheme of possible transitions for $\gamma < {}^4/_3$. In this case the cyclic repetition of the link $M_{1,-1} \to N_{1,-1}$, i.e. a quasi-periodic process of transitions between the states of small and large ellipticity, is possible during the contraction.

We illustrate the behavior of the model in question using the numerical examples. Consider an ellipsoid of revolution rotating or not rotating about one of the axes (longitudinal axis of the ellipsoid). In this case the matrix contains only three distinct elements: $F_{11} = F_{22}$, F_{33}, $F_{12} = -F_{21}$. We write the system of differential equations for these elements in the form

$$\ddot{F}_{11} = \alpha_1 \frac{F_{22}F_{33}}{\varphi^\gamma} + \alpha_2 \int_0^\infty \frac{F_{11}ds}{(F_{11}^2 + F_{12}^2 + s)\psi} \tag{7}$$

$$\ddot{F}_{33} = \alpha_1 \frac{F_{11}F_{22} + F_{12}^2}{\varphi^\gamma} + \alpha_2 \int_0^\infty \frac{F_{33}\, ds}{(F_{33}^2 + s)\psi}$$

$$\ddot{F}_{12} = \alpha_1 \frac{F_{11}F_{33}}{\varphi^\gamma} + \alpha_2 \int_0^\infty \frac{F_{12}\, ds}{(F_{11}^2 + F_{12}^2 + s)\psi}$$

$$\varphi = F_{33}(F_{11}F_{22} + F_{12}^2), \quad \psi = [(F_{11}^2 + F_{12}^2 + s)(F_{22}^2 + F_{12}^2 + s)(F_{33}^2 + s)]^{1/2}$$

When rotation is absent, then the behavior of the cloud under a given initial push towards the center and at two different values of the charge density (the density α_2/α_1 changes by 10 times) is shown in Fig.3 as the dependence of the semi-axes on time. Here $\gamma = {}^5/_3$, therefore the case in question corresponds to the first of the transition diagrams given above. Fig.3 demonstrates the passage of the system to the attraction point $M_{1,1}$.

*) See the previous footnote.

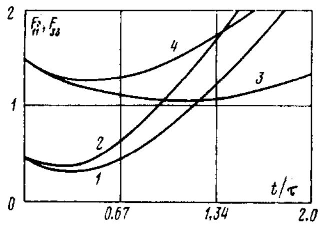

Fig.3

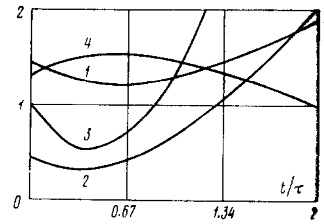

Fig.4

The quantities F_{33} (curves *1* and *2*) and F_{11} (curves *3* and *4*) are plotted along the vertical, and their values characterize the size of the longitudinal and transverse axis, respectively. The passage from the curves *1*, *3* to *2*, *4* takes place with a 10-fold increase in the charge density. We see that the rate of divergence of the axis sizes is greater at the lower charge densities, (a more intense increase in the ellipticity, together with a general increase of the cloud volume). Fig.4 depicts the behavior of gas with rotation, with the parameters corresponding to the curves *1* and *3* of Fig.3. The values of the transverse and longitudinal axes are plotted in dimensionless parameters along the vertical axis (curves *1* and *2*), as well as the ratio of the actual volume of the ellipsoid to its initial value (curve *3*) and the angular velocity of rotation (curve *4*). The transitions of Fig.4 also belong to the first diagram (the chain of transitions $K_{-1} \rightarrow K_1 \rightarrow M_{1,1}$ is almost the same). We note that the rate of growth of the ellipticity is slower than that in the case without rotation.

The properties shown and schemes of transitions given are of interest when considering the problem of inertial compression of plasma. The realisation of one or another transition link is determined by the choice of the initial conditions, and this makes it possible to vary the sequence of transitions of the ellipsoid form. The presence of a charge leads to increased sphericity of the gas cloud. The property of transition from the expansion with small ellipticity to expansion with large ellipticity during the passage from the values of the adiabatic index greater than $^4/_3$ to those smaller than $^4/_3$ is of interest.

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